SET THEORY QUESTIONS IN HOMOTOPY THEORY. LECTURE NOTES. LCMH 2011.

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1. Introduction

This talk introduces those open problems in homotopy theory in which set theory and large cardinals play a role. The following diagrams should be kept in mind; we will explain the meaning and relevance of these collections in the next sections.

$$\{localizing\ subcategories\} \supseteq \{CBCs\} \supseteq \{HBCs\}$$
$$\{localizing\ subcategories\} \supseteq \{singly\ generated\ loc.\ subcats.\} \supseteq \{HBCs\}$$

Question 1.1. The main questions regarding these collections and inclusions are of four general types.

(1) Is it a set or a proper class?
(2) Is the inclusion an equality?
(3) For which categories is this true?
(4) Under what, if any, large-cardinal axioms is this true?

2. Categorical context

(Stable) homotopy theorists classically worked in the stable homotopy category of (p-local) spectra $\mathcal{S}$, but it is now common to generalize to other contexts, which we will briefly mention.

A tensor-triangulated category [HPS97, A.2] is a triangulated category with a closed symmetric monoidal product (the tensor product) that is compatible with the triangulation. We denote the tensor product by $\wedge$, the unit by S, and (graded) morphism sets by $[X, Y]_*$.

A monogenic stable homotopy category [HPS97] is a tensor-triangulated category with arbitrary coproducts, such that the unit S is a small (see below), weak generator (i.e. $[S, X]_* = 0$ implies $X = 0$). Such categories always have arbitrary products, and Brown representability holds (i.e. all cohomology theories are represented). Examples are the category of spectra, the derived category $D(R)$ of a commutative ring R, or the stable module category $\mathrm{StMod}(kG)$.

A Noetherian stable homotopy category is a monogenic stable homotopy category in which the endomorphism ring $[S, S]_*$ is a Noetherian ring. (We also require a

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certain technical hypothesis; see [HPS97, Ch.6]). The primary example is $D(R)$, where R is a commutative Noetherian ring.

Throughout this talk, for simplicity we'll work with the category of p-local spectra $\mathcal{S}$, unless mentioning otherwise. However, most statements can be applied verbatim, or extended, to arbitrary (monogenic) stable homotopy categories, or tensor-triangulated categories, without much extra effort.

There are many explicit constructions of the category $\mathcal{S}$, and many equivalent models for $\mathcal{S}$, but for most of what we describe here, one need only use the above properties. Note that in $\mathcal{S}$ we have representability theorems for both cohomology and homology, so the objects of $\mathcal{S}$ are in bijection with cohomology theories, and in bijection with homology theories.

We need the following definitions.

Definition 2.1. We say an object X is *small* (or *finite*, or *compact*) if $\coprod[X, Y_\alpha] \to [X, \coprod Y_\alpha]$ is always an equivalence, when it exists. The full subcategory of finite objects is essentially small, and is denoted $\mathcal{F}$.

Definition 2.2. Let $\mathcal{C}$ be a triangulated category, and $\mathcal{D}$ be a full subcategory. We say that $\mathcal{D}$ is *thick* if it is closed under suspension, the formation of triangles, and summands (i.e. $X \coprod Y$ in $\mathcal{D}$ implies X and Y in $\mathcal{D}$). We say that $\mathcal{D}$ is *localizing* if it is closed under suspension, triangles and arbitrary coproducts, and *colocalizing* if it is closed under suspension, triangles and arbitrary products.

Let $\mathsf{th}(X)$ (resp. $\mathsf{loc}(X)$) denote the smallest thick (resp. localizing) subcategory containing X. Localizing subcategories of the form $\mathsf{loc}(\mathrm{X})$ are called *singly generated* (or *principal*). In a monogenic stable homotopy category $\mathcal{C}$, like $\mathcal{S}$ or $D(R)$, we have $\mathsf{th}(S) = \mathcal{F}$ and $\mathsf{loc}(S) = \mathcal{C}$.

Question 2.3. Is there a set of localizing subcategories?

Classifying the localizing subcategories, and the thick subcategories of finite objects, is one of the biggest goals in homotopy theory. In 1998, Hopkins and Smith classified the thick subcategories of finite spectra - this remains one of the most beautiful and useful results in stable homotopy theory. Since we don't know if there is a set of them, localizing subcategories are harder to get a handle on. Bousfield classes offer one approach.

3. (Homological) Bousfield classes

Definition 3.1. Let E, F be objects in $\mathcal{S}$. The *(homological) Bousfield class* of E is

$$\langle E \rangle = \{W \mid W \wedge E = 0\}.$$

If $W \wedge E = 0$, we say W is *E-acyclic*. We say E and F are *Bousfield equivalent* if $\langle E \rangle = \langle F \rangle$; this is an equivalence relation.

Question 3.2. Is there a set of homological Bousfield classes (HBCs)?

The answer is known to be yes, in most contexts. Okhawa [Ohk89] gave this result in spectra; Dwyer and Palmieri [DP01] for "Brown categories"; Hovey, Palmieri, and Strickland [HPS97, Ch.6] for Noetherian stable homotopy categories; and recently Iyengar and Krause [IK11] showed this is true in $D(R)$ for any commutative ring R.

There is a partial ordering on HBCs, given by reverse inclusion. The coproduct gives a join operation $\vee_\alpha\langle X_\alpha\rangle = \langle \vee_\alpha X_\alpha\rangle$; there is a maximum element $\langle S\rangle$ and a minimum element $\langle 0\rangle$. Because there is a set of Bousfield equivalence classes, and a minimum, we can define a meet operation (as the join of the *set* of lower bounds). This makes the collection of HBCs into a complete lattice, called the *Bousfield lattice*.

The Bousfield lattice (BL) for spectra has been studied extensively [Bou79a, Bou79b, HP99]. The BL of a Noetherian stable homotopy category was studied in [HPS97]; the BL of the derived category $D(\Lambda)$, where Λ is a particular non-Noetherian ring, in [DP08]; and the BL of some general tensor-triangulated categories in [IK11].

It is not hard to see that every HBC is closed under triangles and coproducts, hence is a localizing subcategory. Because so much is known about the Bousfield lattice, one of the most important open problems is the following.

Question 3.3. Is every localizing subcategory an HBC?

In a Noetherian stable homotopy category, the answer is yes [HPS97]. In spectra we have the following lemma, which is [HP99, Prop. 9.2].

Lemma 3.4. *The following are equivalent.*

1. *Every localizing subcategory is an HBC.*
2. *Every singly generated localizing subcategory is an HBC.*
3. *For every X, $\mathsf{loc}(X) = \langle aX\rangle$.*
4. *$\langle X\rangle \le \langle Y\rangle$ if and only if $X \in \mathsf{loc}(Y)$.*

The objects aX were defined by Bousfield [Bou79b, 1.13], in order to get a nice complementation operation a(-) on HBCs, given by $a\langle X\rangle := \langle aX\rangle$. His construction also gives $\langle X\rangle = \mathsf{loc}(aX)$, which shows that every HBC is a singly generated localizing subcategory.

Bousfield's work uses transfinite induction, and predates many relevant developments in the field. It seems plausible that, perhaps accepting some large-cardinal axiom, it may be possible to make progress on item (3) above.

This lemma applies to spectra, but in fact would apply to any category with a set of HBCs and a good $a(-)$ construction. Constructing $a(-)$ in $D(R)$, for example, via transfinite induction, should be within reach.

We will describe the dual picture, of cohomological Bousfield classes, below. But first, we discuss the relevance of these concepts to localization. It is here that large-cardinal axioms appear.

4. Localization

Definition 4.1. A full subcategory $\mathcal{L}$ of a category $\mathcal{T}$ is *reflective* if the inclusion $\mathcal{L} \hookrightarrow \mathcal{T}$ has a left adjoint $\mathcal{T} \to \mathcal{L}$. Then the composite $L : \mathcal{T} \to \mathcal{T}$ is called a *reflection* or *localization* onto $\mathcal{L}$.

The objects X such that $LX = 0$ are called L-*acyclic*, and the objects in the image $\mathcal{L}$ of L are called L-*local*. A morphism $f : X \to Y$ is called an L-*equivalence* if Lf is an isomorphism.

A full subcategory $\mathcal{C}$ of $\mathcal{T}$ is *coreflective* if the inclusion $\mathcal{C} \hookrightarrow \mathcal{T}$ has a right adjoint. The composite $C : \mathcal{T} \to \mathcal{T}$ is called a *coreflection* or *colocalization* onto $\mathcal{C}$.

There is a bijection between localization and colocalization functors, so that a subcategory is coreflective if and only if it is the collection of acyclics for some localization functor [CGR11, HPS97].

As mentioned above, every object E in $\mathcal{S}$ determines a homology functor, $E_*(X) = [S, X \wedge E]_*$, and every homology functor can be represented in this way by an object in $\mathcal{S}$. Because $[S, X \wedge E]_* = 0$ if and only if $X \wedge E = 0$, we see that

$$\langle E \rangle = \{W \mid E_*(W) = 0\}.$$

Historically, the following important result initiated the use of localization functors in homotopy theory.

Theorem 4.2. (Bousfield 1979) *Given a homology theory E_* on $\mathcal{S}$, there exists a localization functor, L_E, whose acyclics are precisely $\langle E \rangle$.*

We paraphrase this by saying that "homological localizations exist." This was essentially a set theory issue. It was known that one could construct a large category that inverted any set of morphisms [GZ67], however the resulting category would not necessarily have morphism sets.

Note that a localization functor L is determined by its class of L-acyclics (or by its class of L-locals, or by its L-equivalences), and so $\langle E \rangle = \langle F \rangle$ if and only if localization L_E is the same as localization L_F. In this sense, the Bousfield lattice is a description of all the different localizations that arise from homology functors.

Recently, much interesting work has been done in answering a more general question.

Question 4.3. Given a localizing subcategory $\mathcal{D}$, when is there a localization functor whose acyclics are precisely $\mathcal{D}$? In other words, are all localizing subcategories coreflective?

Casacuberta, Gutiérrez, and Rosický have given an answer that depends on Vopěnka's principle, a large-cardinal axiom.

Theorem 4.4. [CGR11, Thm. 3.9] *Let $\mathcal{C}$ be a stable combinatorial model category. If Vopěnka's principle holds, then every localizing subcategory in $Ho(\mathcal{C})$ is singly generated and coreflective.*

This applies to the category of spectra (using simplicial sets as a model), and to the derived category $D(R)$ of a commutative ring R. There is also a slightly weaker result, that does not require Vopěnka's principle.

Theorem 4.5. [CGR11, Prop. 3.7] *Let $\mathcal{C}$ be a stable combinatorial model category. Then every singly generated localizing subcategory of $\mathcal{C}$ is coreflective.*

(Both of these results actually apply more generally to *semilocalizing* subcategories, in which we drop the requirement that the subcategory be closed under fibres.)

Question 4.6. To what extent can the large-cardinal assumption be weakened in these results?

For example, we know that ZFC is sufficient when localizing at a homological Bousfield class. As we'll discuss below, recent work [BCMR11] has shown that one can localize at a cohomological Bousfield class with assumptions that are weaker than Vopěnka's principle.

5. Cohomological Bousfield Classes

Recall that every object E determines a cohomology functor $E^*(X) = [X, E]_*$, and every cohomology functor arises in this way.

Definition 5.1. The *cohomological Bousfield class* (CBC) of E is

$$\langle E^* \rangle = \{X \mid [X, E]_* = 0\}.$$

We say that E and F are *cohomologically Bousfield equivalent* if $\langle E^* \rangle = \langle F^* \rangle$.

This gives an equivalence relation. There is a partial ordering on CBCs, again given by reverse inclusion. There is a minimum $\langle 0^* \rangle$, a maximum $\langle (IS)^* \rangle$ (where $I(-)$ is a Brown-Comenetz functor), and a join operation. Note that $\langle E^* \rangle$ is also closed under triangles and coproducts, so every CBC is a localizing subcategory.

Question 5.2. Is there a set of cohomological Bousfield classes?

Unlike in the HBC case, as yet we have no answer to this question. So there is no "cohomological Bousfield lattice." Having a set of CBCs would also allow for several well-behaved adjoint maps between the collection of HBCs and the collection of CBCs, that would help us to understand the relationship between these collections.

Cohomological Bousfield classes are important because they are a generalization of HBCs.

Lemma 5.3. [Hov95] *Every HBC is a CBC.*

In particular, $\langle X \rangle = \langle (IX)^* \rangle$. Hovey gave this result for $\mathcal{S}$, but it holds in any category with a good Brown-Comenetz functor $I(-)$, including Noetherian stable homotopy categories and several examples of derived categories of non-Noetherian rings.

In fact, Hovey conjectures the converse as well.

Question 5.4. Is every CBC an HBC?

There is some evidence for this in the category of spectra. For many important spectra - $K(n), E(n), KO, KT, Ell$ - we have $\langle X^* \rangle = \langle X \rangle$. Hovey shows that for all finite spectra F, $\langle F^* \rangle$ is an HBC, and likewise for all spectra of finite type.

Recently, we've shown that $\langle X^* \rangle = \langle X \rangle$ for all X in a Noetherian stable homotopy category, such as the derived category of a commutative Noetherian ring.

A positive answer to this question would also answer the previous question, and would be evidence towards Question 3.3. A negative answer would give a negative answer to Question 3.3, and promote the importance of CBCs in understanding localizing subcategories. Along these lines, we have the next question.

Question 5.5. Is every localizing subcategory a CBC?

Large-cardinal axioms may be relevant here, due to the following lemma.

Lemma 5.6. *If all colocalizing subcategories are singly generated, then every localizing subcategory is a CBC.*

Recall that, according to Theorem 4.3, Vopěnka's principle implies that all localizing subcategories are singly generated in $\mathcal{S}$.

We conclude with some comments about cohomological localization. Unlike in the homological case (Theorem 4.1), it is unknown if cohomological localizations exist in ZFC. Since all CBCs are localizing subcategories, Theorem 4.3 says that cohomological localizations exist under the assumption of Vopěnka's principle. By looking at the complexity of the formulas used to define CBCs, [BCMR11] shows that, in fact, cohomological localization in $\mathcal{S}$ follows from the existence of sufficiently large supercompact cardinals - an assumption that is weaker than Vopěnka's principle. However, if the complexity of the CBC definition is shown to be strictly greater than that of HBCs, this may imply that ZFC is insufficient for the existence of cohomological localization.

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